

Limited Frequency Range Observations of Cosmological Point Sources

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ABSTRACT

This paper advances a general proposal for testing non-standard cosmological models by means of observational relations of cosmological point sources in some specific waveband, and their use in the context of the data provided by the galaxy redshift surveys, but for any cosmological metric. By starting from the general theory for observations in relativistic cosmology the equations for colour, K-correction, and number counts of cosmological point sources are discussed in the context of curved spacetimes. The number counts equation is also written in terms of the selection and luminosity functions, which provides a relativistic generalization of its Euclidean version. Since these observables were not derived in the framework of any specific cosmology, they are valid for *any* cosmological model. The hypotheses used in such derivation are reviewed, together with some difficulties for the practical use of those observables.

Introduction

The standard Friedmann models are generally considered as the best approximation for the observed large scale distribution of galaxies, since the results predicted by these models are usually quite good approximations of observations.¹ However, although no observational evidence was so far found to severely contradict this widespread belief, the question remains of whether or not other cosmological models could also provide theoretical predictions in line with observations. This is obviously an important aspect in the general acceptance of the standard Friedmannian models as good approximations of the observed Universe, inasmuch as we can only have a direct response to the question of how good the Friedmann models really are, if we are able to test the data against the predictions of other non-standard cosmological models.

Nevertheless, cosmography is presently dominated by observational relations derived only within the Friedmannian context,^{1–4} and obviously those relations do not allow comparisons between standard and non-standard cosmologies. Therefore, in practice we presently have a situation where the observational test of non-standard models is quite difficult due to the absence of detailed and observationally-based relations derived for that purpose.

There are exceptions, however, and the basis of a general theory for observations of cosmological sources was presented by George Ellis,⁵ although later, in a series of papers,^{6–8} the theory was further developed, with the presentation of detailed calculations of observational relations from where cosmological effects can be identified and separated from the brightness profile evolution of the sources.

Although such study was a step forward in the possibility of direct observational test of non-standard cosmological models, this detailed theory^{6–8} equally demands detailed observations of the sources, a task usually not feasible when dealing with large scale redshift surveys, where the total number of observed objects varies from hundreds to thousands of galaxies. Actually, often it is not even desirable to obtain such detailed observations since what is often being sought are data for doing statistics on the distribution of galaxies.

The approach of this work differs from those quoted above because in here cosmological sources are considered point sources, and therefore observables like flux and colour are integrated over the whole object. This is a reasonable approximation for objects included in these surveys, since they are usually so faint that very detailed observations of their structure are still difficult with the presently available techniques. Therefore, by treating galaxies as point sources we can, at least in principle, apply the methods presented in this paper to the large and deep galaxy surveys presently available.

The observational relations discussed here were derived with the aim of comparing with this redshift surveys of galaxies. As a consequence, the theory used here offers the possibility of comparing the predictions of different cosmological models with the need of much less real data than demanded by the detailed theory mentioned above.^{6–8} Besides, this simpler view of the problem creates the option of a first order test of cosmological models against observations without the need of detailed data, which in turn would demand a more complex and demanding analysis. However, in order to be able to obtain observational relations capable of being compared with observations, to a certain extent we need to depart from the basic approach⁵ and discuss in detail some specific observations in cosmology within some specific bandwidth, since this is the way astronomers deal with their data.

This paper is the first of a series in a programme for investigating whether or not other, non-standard, cosmological models could also explain the data obtained from the large-scale redshift surveys of galaxies. Here I shall review the basic theory for observational relations in limited frequency bandwidth, and the quantities which are mostly used by observers. In doing so I will put together some basic results which will form the common ground from where the general approach of this proposed research programme should start. I will also extend some aspects of this theory, particularly the number-counts expression, and indicate where the connection among these observational quantities, real astronomical observations, and the spacetime geometry takes place. In short, such a connection appears when the observables are written in terms of the redshift and the cosmological distances, since both can only be explicitly written when a spacetime metric is assumed. Even when

all observables are written solely in terms of the redshift, this connection will appear in the functional form between the observational quantities and the redshift, as this functional relationship is dependable on the chosen spacetime geometry.

Basic Definitions and Equations

Let us call F the *bolometric flux* as measured by the observer. This is the rate at which radiation crosses unit area per unit time in all frequencies. Then F_G will be the *bolometric galaxy flux* measured across an unit sphere located in a locally Euclidean space at rest with a galaxy or a cosmological source.⁵

The distance definitions used here are three: *i*) the *observer area distance* r_0 is the area distance of a source as measured by the observer; *ii*) the *galaxy area distance* r_G is defined as the area distance to the observer as measured from the distant galactic source. This quantity is unobservable, by definition; *iii*) the *luminosity distance* d_ℓ is the distance measured by the observer as if the space were flat and non-expanding, that is, as if the space were stationary and Euclidean. The observer area distance r_0 is also called *angular diameter distance*,² and *corrected luminosity distance*.¹¹ The galaxy area distance r_G is also named *effective distance*,¹² *angular size distance*,¹ *transverse comoving distance*,¹³ and *proper motion distance*.¹⁴ These three definitions of distance are related to each other by *Etherington's reciprocity theorem*,^{5,15,16}

$$d_\ell = r_0 (1 + z)^2 = r_G (1 + z), \quad (1)$$

where z is the *redshift* of the source. All these distances tend to the same Euclidean value as $z \rightarrow 0$, but greatly differ at large redshift.^{9,10}

Although the equation above appears in standard texts of observational cosmology, with very few exceptions^{5,16} they all fail to acknowledge the generality of the theorem, and give due credit to Etherington's 1933 discovery. The reciprocity relation was proven for general null geodesics, without specifying *any* metric, and, therefore, it is *not at all* restricted to standard cosmologies.

Let us now call L the *bolometric source luminosity*, that is, the total rate of radiating energy emitted by the source and measured through an unity sphere located in a locally Euclidean spacetime near the source. Then ν will be the *observed frequency* of the radiation, and ν_G the *emitted frequency*, that is, the frequency of the same radiation ν received by the observer, but at rest-frame of the emitting galaxy.

The *source spectrum function* $\mathcal{J}(\nu_G)$ gives the proportion of radiation emitted by the source at a certain frequency ν_G as measured at the rest frame of the source. This quantity is a property of the source, giving the percentage of emitted radiation, and obeying the normalization condition, $\int_0^\infty \mathcal{J}(\nu_G) d\nu_G = 1$. Then $L_{\nu_G} = L \mathcal{J}(\nu_G)$ is the *specific source luminosity*, giving the rate at which radiation is emitted by the source at the frequency ν_G

at its locally Euclidean rest frame. To summarize, we have that,

$$L = \int_{2\text{-sphere}} F_G dA = 4\pi F_G = \int_0^\infty L_{\nu_G} d\nu_G = \int_0^\infty L \mathcal{J}(\nu_G) d\nu_G. \quad (2)$$

The redshift z is defined by,

$$1 + z = \frac{\lambda_{\text{observed}}}{\lambda_{\text{emitted}}} = \frac{\nu_G}{\nu}, \quad (3)$$

and from the expressions above it follows that

$$d\nu = \frac{d\nu_G}{(1+z)}, \quad (4)$$

and

$$F = \frac{F_G}{(r_0)^2(1+z)^4} = \frac{F_G}{(r_G)^2(1+z)^2} = \frac{F_G}{(d_\ell)^2}. \quad (5)$$

The connection of the model with the spacetime geometry appears in the expressions for the redshift and the different definitions of distance. That can be seen if we remember that in the general geometric case the redshift is given by,⁵

$$1 + z = \frac{(u^a k_a)_{\text{source}}}{(u^a k_a)_{\text{observer}}}, \quad (6)$$

where u^a is the observer's four-velocity, and k^a is the tangent vector of the null geodesic connecting source and observer, that is, the past light cone. This expression allows us to calculate z for any given spacetime geometry. If we assume that source and observer are comoving, then $u^a = \delta_0^a$ implies that $u^b k_b = k^b g_{0b}$, and the redshift may be rewritten as,

$$1 + z = \frac{[g_{0b}(dx^b/dy)]_{\text{source}}}{[g_{0b}(dx^b/dy)]_{\text{observer}}}. \quad (7)$$

Here y is the affine parameter along the null geodesics connecting source and observer, and g_{ab} is the metric tensor. Inasmuch as dx^b/dy and g_{ab} can only be determined when a spacetime geometry is defined by some line element dS^2 , the function $g_{0b}(dx^b/dy)$ and, ultimately, the redshift as well are directly dependable on the geometry of the model. Although z is an astronomically observable quantity, its specific internal relationship with other internal cosmological quantities of the model will be set by the metric tensor.

The observer area distance r_0 is defined by,^{5,16}

$$(r_0)^2 = \frac{dA_0}{d\Omega_0}, \quad (8)$$

where dA_0 is the cross-sectional area of a bundle of null geodesics measured *at the source's rest frame*, and diverging from the observer at some point, and $d\Omega_0$ is the solid angle subtended by this bundle. This quantity can in principle be measured if we had *intrinsic* astrophysically-determined dimensions of the source, but it can also be obtained from the assumed spacetime geometry, especially in spherically symmetric metrics, from where it can be easily calculated. For the Einstein-de Sitter cosmology, detailed calculations for many observables can be found elsewhere.^{10,17}

Frequency Bandwidth Observational Relations

Flux and Magnitude

The flux within some specific wavelength range can be obtained if we consider equations (2), (4) and (5). Then we have,⁵

$$F = \frac{\int_0^\infty L \mathcal{J}(\nu_G) d\nu_G}{4\pi (r_0)^2 (1+z)^4} = \frac{L}{4\pi} \frac{\int_0^\infty \mathcal{J}[\nu(1+z)] (1+z) d\nu}{(r_0)^2 (1+z)^4} = \frac{L}{4\pi} \frac{\int_0^\infty \mathcal{J}[\nu(1+z)] d\nu}{(r_0)^2 (1+z)^3}. \quad (9)$$

Therefore, the *specific flux* F_ν measured in the frequency range ν , $\nu + d\nu$ by the observer, may be written as

$$F_\nu d\nu = \frac{L}{4\pi} \frac{\mathcal{J}[\nu(1+z)] d\nu}{(r_0)^2 (1+z)^3}. \quad (10)$$

The apparent magnitude in a specific observed frequency bandwidth is,

$$m_W = -2.5 \log \int_0^\infty F_\nu W(\nu) d\nu + \text{constant}, \quad (11)$$

where $W(\nu)$ is the function which defines the spectral interval of the observed flux (the standard UBV system, for instance). This is a sensitivity function of the atmosphere, telescope and detecting device. Thus, from equations (10) and (11) the apparent magnitude in a specified spectral interval W yields,

$$m_W = -2.5 \log \left\{ \frac{L}{4\pi} \frac{1}{(r_0)^2 (1+z)^3} \int_0^\infty W(\nu) \mathcal{J}[\nu(1+z)] d\nu \right\} + \text{constant}. \quad (12)$$

Since cosmological sources do evolve, the intrinsic luminosity L changes according to the evolutionary stage of the source, and therefore, L is actually a function of the redshift: $L = L(z)$. Hence, in order to use equation (12) to obtain the apparent magnitude evolution of the source, some theory for luminosity evolution is also necessary. For galaxies, $L(z)$ is usually derived taking into consideration the theory of stellar evolution, from where some simple equations for luminosity evolution can be drawn.^{1,18} Note that equation (12) also indicates that the source spectrum function \mathcal{J} might evolve and change its functional form at different evolutionary stages of the source. In addition, as $\mathcal{J}[\nu(1+z)]$ is a property of the source at a specific redshift, this function must be known in order to calculate the apparent magnitude, unless the K-correction approach is used (see below). For magnitude limited catalogues, the luminosity distance and the observer area distance have both an upper cutoff, which is a function of the apparent magnitude, the frequency bandwidth used in the observations and the luminosity of the sources.

K-Correction

The relations above demand the knowledge of both the source spectrum and the redshift. However, when the source spectrum is not known, it is necessary to introduce a correction

term in order to obtain the bolometric flux from observations. This correction is known as the K-correction, and it is a different way for allowing the effect of the source spectrum.

In deriving the K-correction,^{3,4,19,20} I start by calculating the difference in magnitude produced by the bolometric flux F and the flux F_W measured by the observer, but at the bandwidth $W(\nu)$ in any redshift z . Since,

$$F = \int_0^\infty F_\nu d\nu, \quad F_W = \int_0^\infty F_\nu W(\nu) d\nu, \quad (13)$$

the difference in magnitude $\Delta m(z)$ will be given by

$$\log \frac{F(z)}{F_W(z)} = 0.4\Delta m(z). \quad (14)$$

The rate between the observed flux $F_W(z)$ at a given redshift and at $z = 0$ defines the K-correction. Then, considering equation (14), we have that

$$\frac{F_W(z)}{F_W(0)} = \frac{F(z)}{F(0)} 10^{-0.4K_W}, \quad (15)$$

where we have defined

$$K_W \equiv \Delta m(z) - \Delta m(0). \quad (16)$$

Then it follows that

$$K_W = m_W - m_{\text{bol}} - \Delta m(0), \quad (17)$$

which means that once we know the K-term and the observed magnitude m_W , the bolometric magnitude is known within a constant $\Delta m(0)$. If we now substitute equation (10) into equation (15), and assume $L(z) = L(0)$, it is easy to show that

$$K_W(z) = 2.5 \log \left\{ \frac{\int_0^\infty W(\nu) \mathcal{J}(\nu) d\nu}{\int_0^\infty W(\nu) \mathcal{J}(\nu_G) d\nu_G} \right\}. \quad (18)$$

Remembering that by equation (4) we know that we can have the source spectrum transformed from the rest frame of the source to the rest-frame of the observer by a factor of $(1+z)$, that is, $\mathcal{J}[\nu(1+z)] d\nu = [\mathcal{J}(\nu_G) d\nu_G] / (1+z)$, then we may also write equation (18) as

$$K_W(z) = -2.5 \log(1+z) + 2.5 \log \left\{ \frac{\int_0^\infty W(\nu) \mathcal{J}(\nu) d\nu}{\int_0^\infty W(\nu) \mathcal{J}[\nu(1+z)] d\nu} \right\}. \quad (19)$$

Note that the equations above allow us to write theoretical K-correction expressions for any given spacetime geometry, provided that the line element dS^2 is known beforehand. As a final remark, it is obvious that if the source spectrum is already known, all relevant observational relations can be calculated without the need of the K-correction.

Colour

With the expressions above we can obtain the theoretical equation for the colour of the sources for any given spacetime. Let us consider two bandwidths W and W' . From equation (12) we can find the difference in apparent magnitude for these two frequency bands in order to obtain an equation for the colour of the source in a specific redshift. Let us call this quantity $C_{WW'}$. Thus,

$$C_{WW'}(z) \equiv m_W - m_{W'} = 2.5 \log \left\{ \frac{\int_0^\infty W'(\nu) \mathcal{J}[\nu(1+z)] d\nu}{\int_0^\infty W(\nu) \mathcal{J}[\nu(1+z)] d\nu} \right\}. \quad (20)$$

Considering that cosmological sources do evolve, they should emit different luminosities in different redshifts due to the different evolutionary stages of the stellar contents of the sources, and this is reflected in the equation above by the source spectrum function which may be different for different redshifts. Note, however, that in the equation above the source is assumed to have the same bolometric luminosity in a specific redshift and, therefore, we can only use equation (20) to compare observation of objects of the same class and at similar evolutionary stages in certain z , since $L = L(z)$. This often means galaxies of the same morphological type. In other words, equation (20) assumes that a homogeneous populations of cosmological sources do exist, and hence, the evolution and structure of the members of such a group will be similar.

Equation (20) also gives us a method for assessing the possible evolution of the source spectrum. For instance, by calculating $(B - V)$ and $(V - R)$ colours for E galaxies with modern determinations of the K-correction, it has been reported⁴ that no colour evolution was found to at least $z = 0.4$. However, for $z \geq 0.3$ it was found that rich clusters of galaxies tend to be bluer (the Butcher-Oemler effect) than at lower redshifts.^{1,21} Therefore, if we start from a certain metric, we can calculate the theoretical redshift range where colour evolution would be most important for the assumed geometry of the cosmological model. Then, assessing evolution could be done by means of multicolour observations. As the luminosity and area distances must be the same in all wavelengths for each given source, if the luminosity-redshift plot is not the same in two colours, this shows that these two colours have different evolution functions. Applications of this idea for searching inhomogeneities, by means of the Lemaître-Tolman-Bondi cosmology, can be found in the literature.²²

Another point worth mentioning, from equation (20) we see that colour is directly related to the intrinsic characteristics of the source, its evolutionary stage, as given by the redshift and the assumptions concerning the real form of the source spectrum function at a certain z . However, this reasoning is valid for point sources whose colours are integrated and, therefore, we are not considering here structures, like galactic disks and halos, which in principle may emit differently and then will produce different colours. If we remember that cosmological sources are usually far enough to make the identification and observation of source structures an observational problem for large scale galaxy surveys, this hypothesis

seems reasonable at least as a first approximation.

Finally, it is clear that in order to obtain a relationship between apparent magnitude and redshift we need some knowledge about the dependence of the intrinsic bolometric luminosity L and the source spectrum function \mathcal{J} with the redshift. It seems that such a knowledge must come from astrophysically independent theories about the intrinsic behaviour and evolution of the sources, and not from the assumed cosmological model.

Number Counts

In any cosmological model if we consider a small affine parameter displacement dy at some point P on a bundle of past null geodesics subtending a solid angle $d\Omega_0$, and if n is the number density of radiating sources per unit proper volume at P, then the *number of sources* in this section of the bundle is,⁵

$$dN = (r_0)^2 d\Omega_0 [n(-k^a u_a)]_P dy, \quad (21)$$

where k^a is the propagation vector of the radiation flux. Equation (21) assumes the counting of *all* sources at P with number density n . Consequently, if we want to consider the more realistic situation that only a fraction of galaxies in the proper volume $dV = (r_0)^2 d\Omega_0 dl = (r_0)^2 d\Omega_0 (-k^a u_a) dy$ is actually detected and included in the observed number count, we have to write dN in terms of a *selection function* ψ which represents this detected fraction of galaxies. Then equation (21) becomes²³

$$dN_0 = \psi dN = \psi [n dV]_P = (r_0)^2 \psi d\Omega_0 [n(-k^a u_a)]_P dy, \quad (22)$$

where dN_0 is the *fractional number of sources actually observed* in the unit proper volume dV with a total of dN sources.

In principle ψ can be estimated from a knowledge of the galactic spectrum, the observer area distance, the redshift, and the detection limit of the sample as given by the limiting flux in a certain frequency bandwidth. The other quantities in equation (22) come from the assumed cosmological model itself.

In order to determine ψ we need to remember that in any spacetime geometry the observed flux in bandwidth W is given by equations (10) and (13),

$$F_w = \frac{L(z)}{4\pi(r_0)^2(1+z)^3} \int_0^\infty W(\nu) \mathcal{J}[\nu(1+z)] d\nu. \quad (23)$$

Then, if a galaxy at a distance r_0 is to be seen at flux F_w , its luminosity $L(z)$ must be bigger than $\{4\pi(r_0)^2(1+z)^3 F_w\} / \{\int_0^\infty W(\nu) \mathcal{J}[\nu(1+z)] d\nu\}$. Therefore, the probability that a galaxy at a distance r_0 and with redshift z is included in a catalog with maximum flux F_w is,

$$\mathcal{P} \propto \psi(\ell) = \int_\ell^\infty \phi(w) dw, \quad (24)$$

where this integral's lower limit is

$$\ell = \frac{1}{L_*} \frac{4\pi(r_0)^2(1+z)^3 F_W(z)}{\int_0^\infty W(\nu) \mathcal{J}[\nu(1+z)] d\nu}, \quad (25)$$

L_* is a parameter, and $\phi(w)$ is the *luminosity function*.¹ In Schechter²⁴ model L_* is a *characteristic luminosity* at which the luminosity function exhibits a rapid change in its slope. Now, if we assume spherical symmetry, then equation (22) becomes,

$$dN_0 = 4\pi(r_0)^2 \psi(\ell) [n(-k^a u_a)]_P dy. \quad (26)$$

Thus, the number of galaxies observed up to an affine parameter y at a point P down the light cone, may be written as

$$N_0 = 4\pi \int_0^y (r_0)^2 \psi(\ell) [n(-k^a u_a)]_P d\bar{y}, \quad (27)$$

which generalizes Peebles' Euclidean equation (7.40)¹ into a relativistic setting.

Equation (27) is deceptively simple. It is in fact a highly non-linear and difficult-to-compute function, as all quantities entering the integrand are functions of the past null cone affine parameter y . Therefore, in principle, they must be explicitly calculated before they can be entered into equation (27). In some cases one may avoid this explicit determination and use instead the radial coordinate,^{10,17,25–28} a method which turns out to be easier than finding these expressions in terms of y . Then, once $N_0(y)$ is obtained, it becomes possible to relate it to other observables, since they are all function of the past null cone affine parameter. For example, if one can derive an analytic expression for the redshift in a given spacetime, say $z = z(y)$, and if this expression can be analytically inverted, then we can write N_0 as a function of z .

It is important to mention that the local number density n is given in units of proper density and, therefore, in order to take a proper account of the curved spacetime geometry, one must relate n to the local density as given by the right hand side of Einstein's field equations. If, for simplicity, we suppose that all sources are galaxies with a similar rest-mass M_g , then $n = \rho/M_g$.

The discussion above shows that the theoretical determination of N_0 depends critically on the spacetime geometry and the luminosity function ϕ . For the latter, in the Schechter²⁴ model it has the form, $\phi(w) = \phi_* w^\alpha e^{-w}$, where ϕ_* and α are constant parameters. One must not forget that this luminosity function shape was originally determined from *local* measurements,²⁴ and it is still under assessment the possible change of shape and parameters of the luminosity function in terms of evolution,^{29–32} that is, as we go down the light cone.

As a final remark, one must note that gravitational lensing magnification can also affect the counting of point sources, because weak sources with low flux might appear brighter due to lensing magnification. Such an effect will not be discussed here, since its full treatment demands more detailed information about the sources themselves, such as considering them as extended ones, and is considered to be most important for QSO's.¹⁶

Conclusion

In this paper I have advanced a general proposal for testing non-standard cosmological models by means of observational relations of cosmological point sources in some specific waveband, and their use in the context of data provided by the galaxy redshift surveys. I have also shown how the relativistic number-counting equation can be expressed in terms of the selection and luminosity functions, generalizing thus the Euclidean number counts expression into a relativistic setting. The equations for colour, and K-correction were also presented. All expressions obtained here are valid for any cosmological metric, since no specific geometry was assumed in such a derivation. Although these observables can be specialized for a given spacetime geometric, some quantities must come from astrophysical considerations, namely the intrinsic luminosity $L(z)$, the source spectrum function $\mathcal{J}(\nu)$, and the luminosity function $\phi(w)$. These cannot be obtained only from geometrical considerations, which means that the determination of the spacetime structure of universe is a task intrinsically linked to astrophysical considerations and results. Further developments and applications of the general approach discussed here are the subject of a forthcoming paper.³³

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